Weak Measurements

on the ground state of 2-D Anti-Ferromagnetic Heisenberg Model

Kriti Baweja

University of Bonn

Supervisors: Prof. Dr. David J. Luitz & Prof. Dr. Thomas C. Luu

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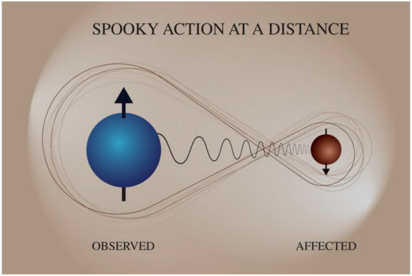
Introduction

Introduction and Motivation

- Fundamental Question: Response of Quantum System to measurement.
- Non-local aspect of measurement has striking manifestation in violation of Bell inequality.
- Causes entanglement transition in hybrid quantum circuits



Grafico/Alamy Stock Vector

Aim: Numerically implement & Study effect of measurement(s) on the ground state of highly-entangled many-body system.

Stochastic Series Expansion Overview

Method

Taylor expand Boltzmann operator

$$\exp(-\beta H) = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} H^n \tag{1}$$

β: inverse temperature, n: expansion order

Partition function:

$$Z = \mathrm{Tr}(\exp(-\beta H)) = \sum_{\{\alpha\}} \langle\alpha| \sum_{n=0}^{\infty} \frac{(-\beta H)^n}{n!} |\alpha\rangle \tag{2}$$

$|\alpha\rangle$: computational basis

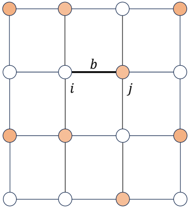
Hamiltonian

Focus: Bipartite, Anti-ferromagnetic Heisenberg Hamiltonian,

$$H = J \sum_{b=1}^{N_b} \boldsymbol{S}_{i(b)} \cdot \boldsymbol{S}_{j(b)} = \sum_b H_b \qquad (3)$$

N_b: number of bonds

$$H = J \sum_b \left(S^z_{i(b)} S^z_{j(b)} + \frac{1}{2} (S^+_{i(b)} S^-_{j(b)} + S^-_{i(b)} S^+_{j(b)}) \right) \qquad (4)$$



2-D Heisenberg Lattice

$$H_b = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Hamiltonian: Simplification

Rewriting H_b using operator with two indices $H_{a,b}$,

$$\begin{aligned} H_{1,b} &= \frac{1}{4}\mathbb{1} - S^z_{i(b)} S^z_{j(b)} \\ H_{2,b} &= \frac{1}{2}(S^+_{i(b)} S^-_{j(b)} + S^-_{i(b)} S^+_{j(b)}) \end{aligned} \tag{5}$$

$$\tilde{H}_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Full Hamiltonian:

$$-\beta H = J \sum_b (H_{1,b} - H_{2,b}) + \frac{J N_b}{4} \mathbb{1} \tag{6}$$

Modification of Z

$$Z = \mathrm{Tr}(\exp(-\beta H)) = \sum_{\{\alpha\}} \langle \alpha | \sum_{n=0}^{\infty} \frac{(-\beta H)^n}{n!} | \alpha \rangle$$

$$(-\beta H)^n = \left(J \sum_b (H_{1,b} - H_{2,b}) \right)^n = \sum_{\{S_n\}} (-1)^{n_2} J^n \prod_{i=0}^{n-1} H_{a(i),b(i)} \tag{7}$$

where n_2 is number of off-diagonal operators. The partition function is:

$$Z = \sum_{\{\alpha\}} \sum_{n=0}^{\infty} \sum_{\{S_n\}} (-1)^{n_2} \frac{(\beta)^n}{n!} \left\langle \alpha \middle| \prod_{p=0}^{n-1} H_{a(p),b(p)} \middle| \alpha \right\rangle \tag{8}$$

Propagated state:

$$|\alpha(p)\rangle \propto \prod_{i=0}^{p-1} H_{a(i),b(i)} |\alpha\rangle \tag{9}$$

Fixed Length Operator String

- n is finite, $n_{max} \propto \beta N$
- Length of operator string L is kept fixed where $L > n$.
- Augment operator string with $L - n$ unit operator, $H_{0,0} = \mathbb{1}$.
- $\binom{L}{n}$ ways to place $H_{0,0}$ in product of L operators.

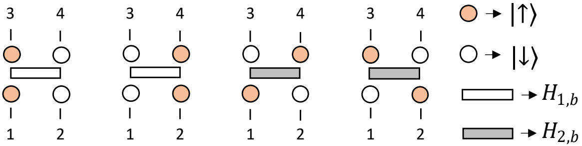
Z in fixed length scheme,

$$Z = \sum_{\{\alpha\}} \sum_{\{S_L\}} \underbrace{(-1)^{n_2} \frac{(\beta)^n (L-n)!}{L!} \left\langle \alpha \left| \prod_{p=0}^{L-1} H_{a(p),b(p)} \right| \alpha \right\rangle}_{W(C=(\alpha, S_L))} \tag{10}$$

Periodicity constraint,

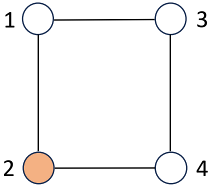
$$|\alpha(L-1)\rangle = |\alpha(0)\rangle \tag{11}$$

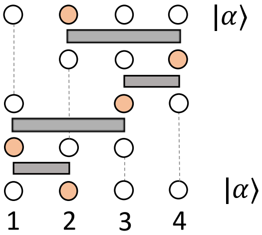
Diagrammatic Representation



Allowed Vertices for $\tilde{H}_b$

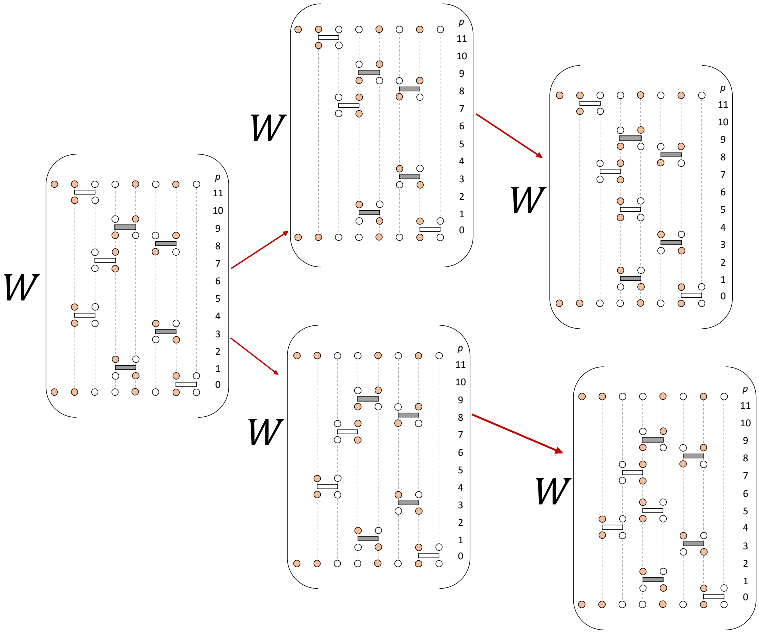
$$Z = \sum_{\{\alpha\}} \sum_{\{S_L\}} (-1)^{n_2} \frac{(\beta)^n (L-n)!}{L!} \left\langle \alpha \middle| \prod_{p=0}^{L-1} H_{a(p),b(p)} \middle| \alpha \right\rangle$$



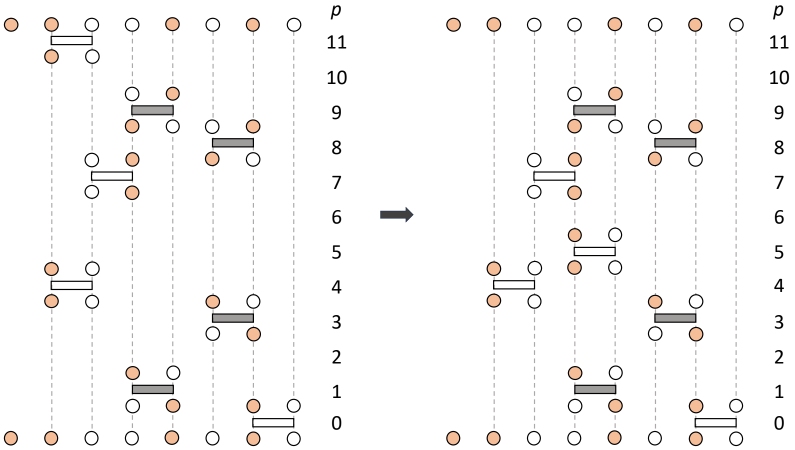


$H_{2,b}$ even for $|\alpha(n)\rangle = |\alpha(0)\rangle$.

Monte Carlo Sampling

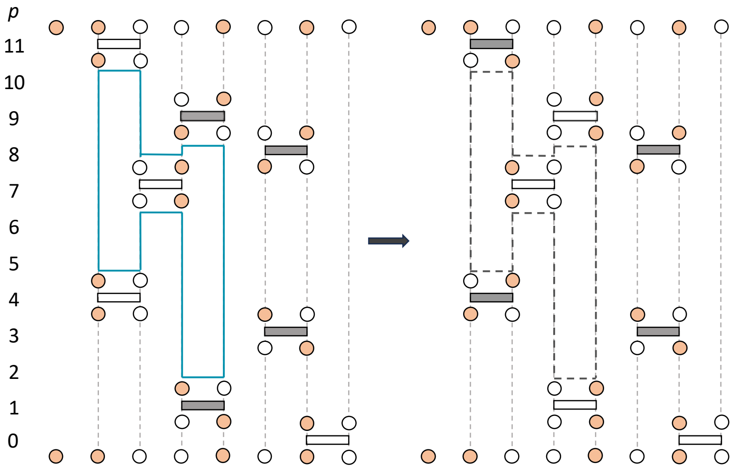


Monte Carlo Updates



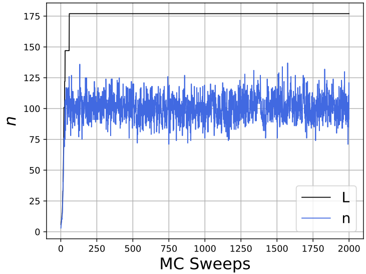
Diagonal Update: $H_{1,b} \leftrightarrow H_{0,0}$

Off-diagonal update

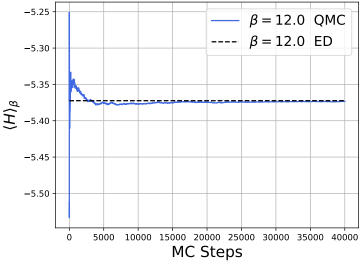


Directed Loop update in SSE: $H_{2,b} \leftrightarrow H_{1,b}$

Results

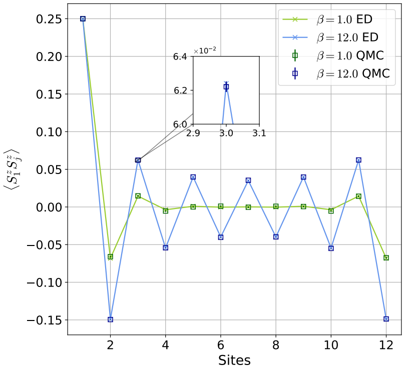


Evolution L and n with Monte Carlo (MC) sweeps during thermalization for 1-D Heisenberg Chain at $\beta = 12.0$.

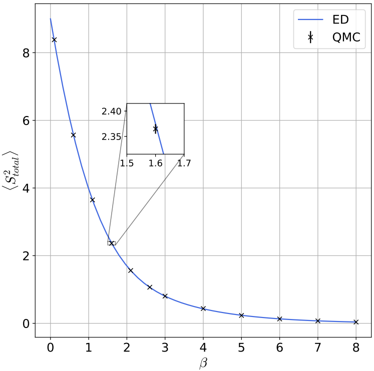


Energy $(-\frac{\langle n \rangle}{\beta} + \frac{JN_b}{4})$ Variation with number of Monte Carlo (MC) steps.

Results Contd.



$\langle S_1^z S_j^z \rangle$ for 1-D Heisenberg Chain with 12 spins



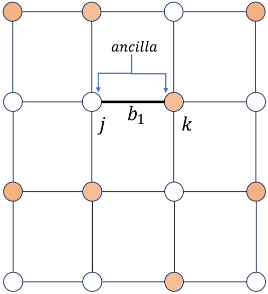
Total Spin ($3 \sum_{i,j} \langle S_i^z S_j^z \rangle$) for 2×6 Heisenberg Lattice.

Introduction to Measurement Scheme

Measurement Scheme

Symmetry preserving measurements:

$$\boldsymbol{S}_j . \boldsymbol{S}_k = -\frac{3}{4} P_{0,jk} + \frac{1}{4} P_{1,jk} \tag{12}$$



Weak measurement scheme

$$P_{0,jk} + P_{1,jk} = \mathbb{1}$$

$P_{0,jk}$: Projection on singlet sector

$$P_{0,jk} = \frac{1}{4} \mathbb{1} - \boldsymbol{S}_j \cdot \boldsymbol{S}_k, \tag{13}$$

$P_{1,jk}$: Projection on triplet sector.

$$P_{1,jk} = \frac{3}{4} \mathbb{1} + \boldsymbol{S}_j \cdot \boldsymbol{S}_k. \tag{14}$$

SSE Implementation of Measurements

Weak Measurement Operator [Weinstein et al, PRB, 2023]

$$Q_{s,jk} = \eta_s^{1/2} e^{\mu(s-\frac{1}{2})\boldsymbol{S}_j\cdot\boldsymbol{S}_k} \tag{15}$$

$s \in \{0,1\}$, η_s: normalization constant, μ:measurement strength

Post-Measurement State:

$$|\Phi_{s_0\ldots s_{N-1}}\rangle \propto \left(\bigotimes_{b_1=0}^{N_{b_1}-1} Q_{s(b_1),j(b_1)k(b_1)}\right)|\Phi\rangle \tag{16}$$

$|\Phi\rangle$: ground state, b_1: measured bonds, $s(b_1)$: measurement type for b_1.

$$Q_{\{s(b_1)\}} := \bigotimes_{b_1} Q_{s(b_1),j(b_1)k(b_1)}$$

$$\rho_{\{s(b_1)\}} = \frac{Q_{\{s(b_1)\}}\, e^{-\beta H}\, Q_{\{s(b_1)\}}}{\mathrm{Tr}(e^{-\beta H} Q^2_{\{s(b_1)\}})} \tag{17}$$

SSE Implementation of Measurements Contd.

Partition function:

$$Z_{\{s(b_1)\}}(\beta,\mu) = \mathrm{Tr}\left(e^{\mu\sum_{b_1}(2s(b_1)-1)\boldsymbol{S}_{j(b_1)}\cdot\boldsymbol{S}_{k(b_1)}} e^{-\beta H}\right) \tag{18}$$

$$M_{\{s\}} := -\sum_{b_1}(2s(b_1)-1)\boldsymbol{S}_{j(b_1)}\cdot\boldsymbol{S}_{k(b_1)}$$

$$M_{\{0\}} = \sum_{b_1}\boldsymbol{S}_{j(b_1)}\cdot\boldsymbol{S}_{k(b_1)} \qquad M_{\{1\}} = -\sum_{b_1}\boldsymbol{S}_{j(b_1)}\cdot\boldsymbol{S}_{k(b_1)} \tag{19}$$

$$Z_{\{s(b_1)\}}(\beta,\mu) = \sum_{m,n=0}^{\infty}\frac{(-\mu)^m(-\beta)^n}{n!m!}\mathrm{Tr}\left(M_{\{s\}}^m H^n\right) \tag{20}$$

where n and m are expansion order for H and $M_{\{s\}}$ respectively.

Simplification of Heisenberg Hamiltonian for Measurement

Triplet Case:

$$M_{1,1,b_1} = \frac{1}{4}\mathbb{1} + S^z_{j(b_1)} S^z_{k(b_1)}, \tag{21}$$

$$M_{1,2,b_1} = \frac{1}{2}(S^+_{j(b_1)} S^-_{k(b_1)} + S^-_{j(b_1)} S^+_{k(b_1)}). \tag{22}$$

$$M_{1,b_1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$-\mu M_{\{1\}} = \sum_{b_1=1}^{N_{b_1}} (M_{1,1,b_1} + M_{1,2,b_1}) + \frac{JN_{b_1}}{4}\mathbb{1} \tag{23}$$

Singlet Case:

$$-\mu M_{\{0\}} = \sum_{b_1=1}^{N_{b_1}} (M_{0,1,b_1} - M_{0,2,b_1}) + \frac{JN_{b_1}}{4}\mathbb{1} \tag{24}$$

Modification of $Z(\beta, \mu)$

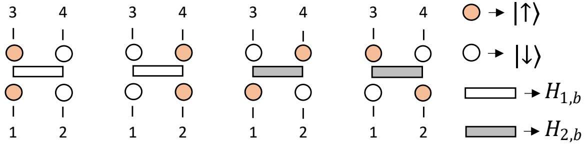
Singlet measurements:

$$Z_{\{0\}}(\beta,\mu) = \sum_{\substack{\{\alpha\},\\ \{S_{L_H}\},\\ \{S_{L_M}\}}} \underbrace{(-1)^{n_2+m_2} \frac{(\mu)^m(\beta)^n(L_H-n)!(L_M-m)!}{L_H!L_M!} \left\langle \alpha \middle| \prod_{p_1=L_H}^{L-1} M_{0,a(p_1),b_1(p_1)} \prod_{p=0}^{L_H-1} H_{a(p),b(p)} \middle| \alpha \right\rangle}_{W_{\{0\}}(C=(\alpha,S_{L_H},S_{L_M}))} \tag{25}$$

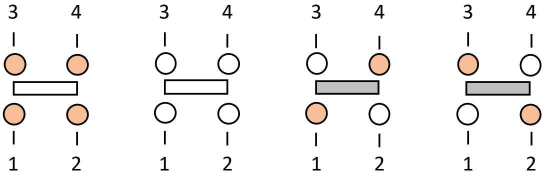
Triplet measurements:

$$Z_{\{1\}}(\beta,\mu) = \sum_{\substack{\{\alpha\},\{S_{L_H}\},\\ \{S_{L_M}\}}} \underbrace{(-1)^{n_2} \frac{(\mu)^m(\beta)^n(L_H-n)!(L_M-m)!}{L_H!L_M!} \left\langle \alpha \middle| \prod_{p_1=L_H}^{L-1} M_{1,a(p_1),b_1(p_1)} \prod_{p=0}^{L_H-1} H_{a(p),b(p)} \middle| \alpha \right\rangle}_{W_{\{1\}}(C=(\alpha,S_{L_H},S_{L_M}))} \tag{26}$$

Allowed Vertices

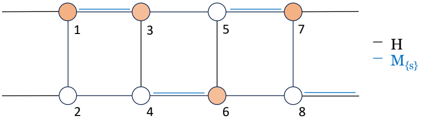


Allowed vertices for $\tilde{H}_b$ & M_{0,b_1}

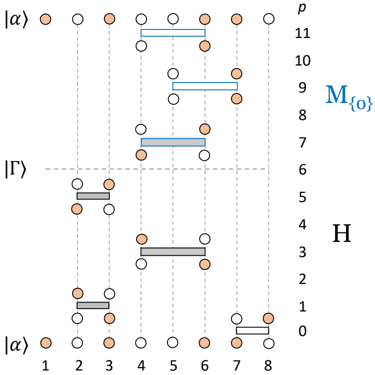


Allowed vertices for M_{1,b_1}

Monte Carlo Sampling



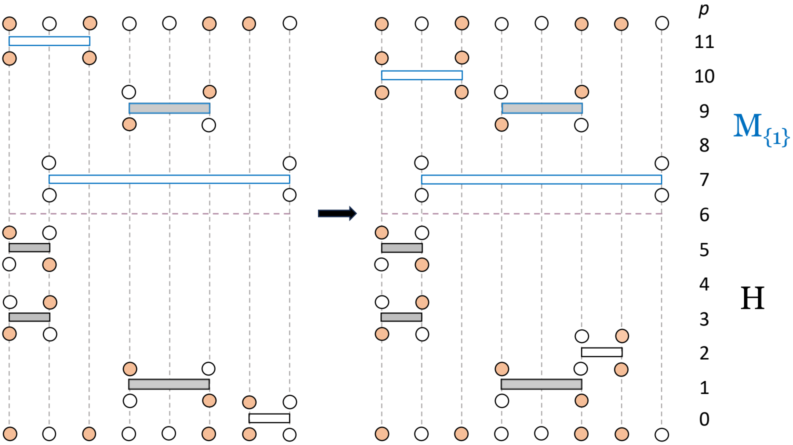
2×6 periodic Heisenberg lattice with measurement



SSE Configuration with $M_{\{0\}}$

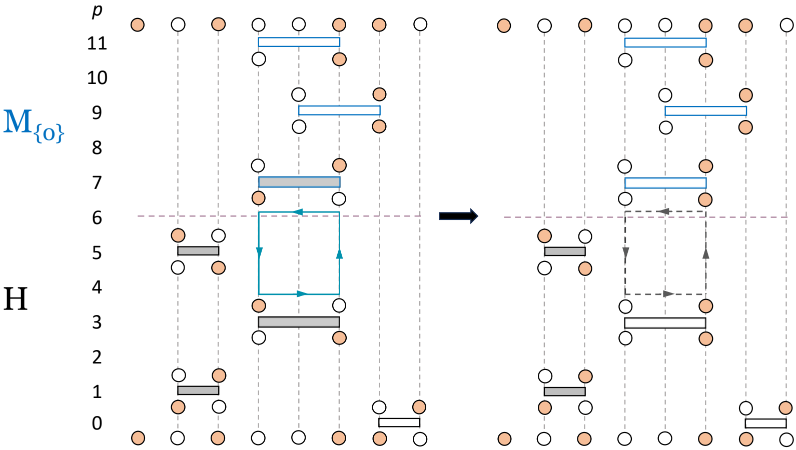
Ergodicity Constraint:
Sample configurations with $|\Gamma\rangle = |\alpha\rangle$ & $|\Gamma\rangle \neq |\alpha\rangle$

Monte Carlo Updates



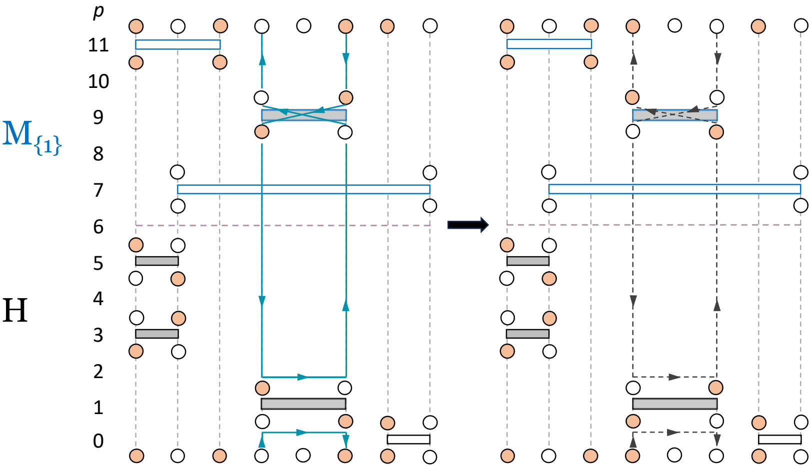
Diagonal update with measurement: $H_{1,b} \leftrightarrow H_{0,0}$ & $M_{1,1,b_1} \leftrightarrow M_{1,0,0}$

Off-diagonal Update $M_{\{0\}}$



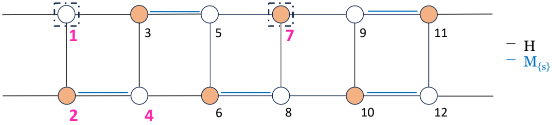
SSE Directed loop update involving $M_{\{0\}}$: $H_{2,b} \leftrightarrow H_{1,b}$ & $M_{0,2,b_1} \leftrightarrow M_{0,1,b_1}$

Off-diagonal Update $M_{\{1\}}$

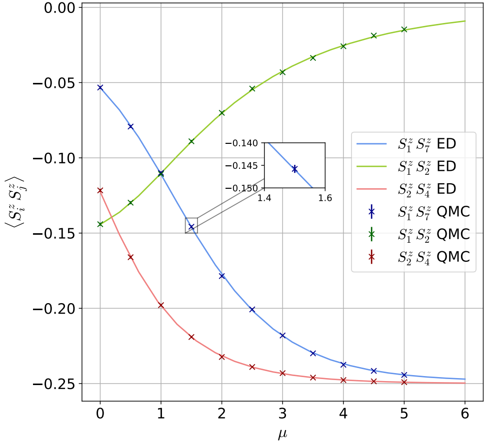


SSE Directed loop update involving $M_{\{1\}}$: $H_{2,b} \leftrightarrow H_{1,b}$ & $M_{1,2,b_1} \leftrightarrow M_{1,1,b_1}$

Singlet Measurements

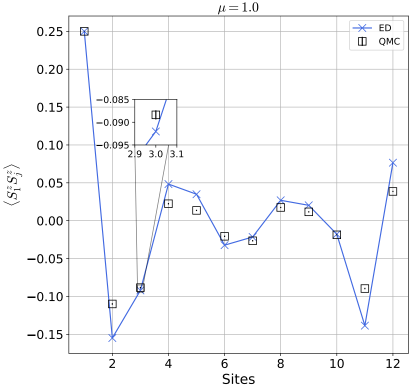


2×6 Heisenberg Lattice with two unmeasured sites

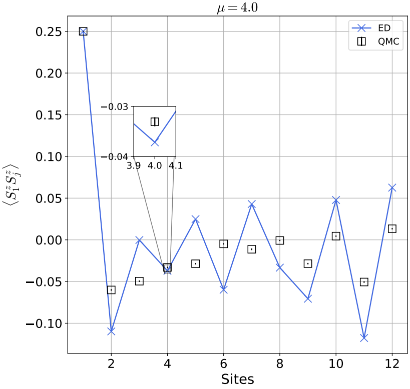


Variation of $\langle S_i^z . S_j^z \rangle$ with μ.

Triplet Measurements

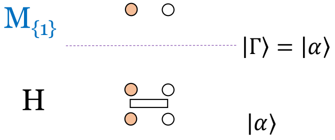


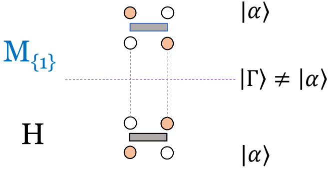
$\langle S_1^z S_j^z \rangle$ for different lattice sites at $\mu = 1.0$



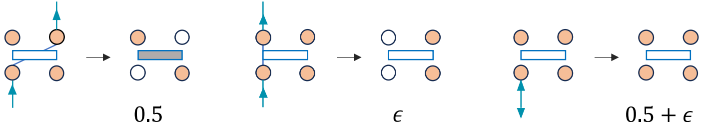
$\langle S_1^z S_j^z \rangle$ for different lattice sites at $\mu = 4.0$

Ergodicity and Sign Problem





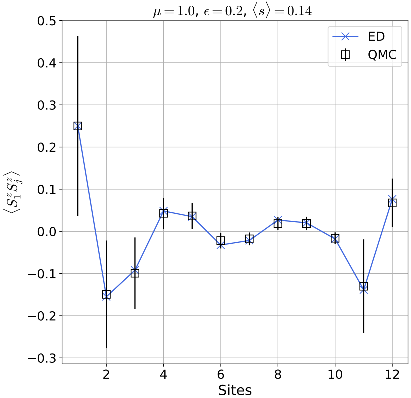
$$\tilde{M}_{1,b_1} = M_{1,b_1} + \epsilon\mathbb{1} = \begin{pmatrix} \frac{1}{2}+\epsilon & 0 & 0 & 0 \\ 0 & \epsilon & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \epsilon & 0 \\ 0 & 0 & 0 & \frac{1}{2}+\epsilon \end{pmatrix}$$

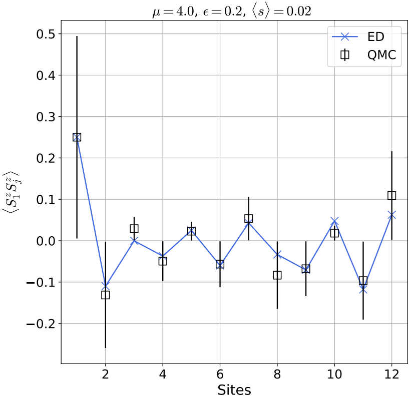


$$P(4|1) = \frac{0.5}{1+2\epsilon} \qquad P(3|1) = \frac{\epsilon}{1+2\epsilon} \qquad P(1|1) = \frac{0.5+\epsilon}{1+2\epsilon}$$

New exit leg rules

Triplet Measurements after Correction

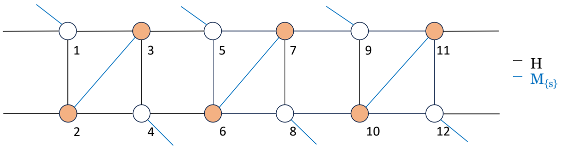




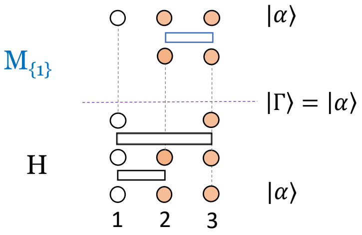
$\langle S_1^z S_j^z\rangle$ for different lattice sites at $\mu = 1.0$,$\epsilon = 0.2$ & $\langle s\rangle = 0.14$

$\langle S_1^z S_j^z\rangle$ for different lattice sites at $\mu = 4.0$, $\epsilon = 0.2$ & $\langle s\rangle = 0.02$

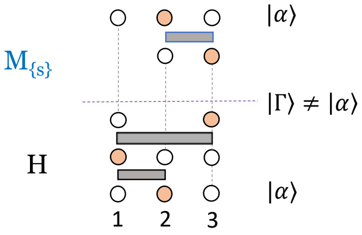
Sign-free Triplet Measurement



Measuring triplets on diagonal bonds

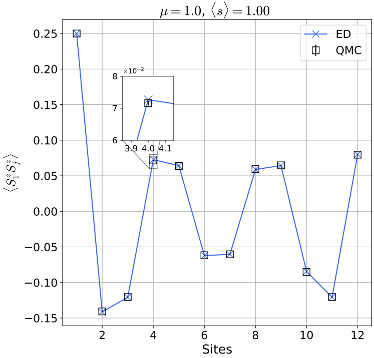


Allowed Configuration for $M_{\{1\}}$

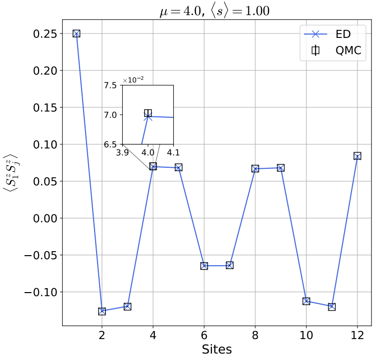


Allowed configuration for $M_{\{s\}}$

Sign-free Triplet Measurement Contd.



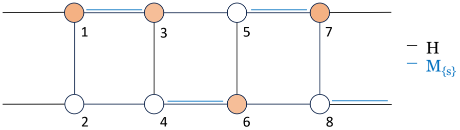
$\langle S_1^z S_j^z \rangle$ for different lattice sites at $\mu = 1.0$

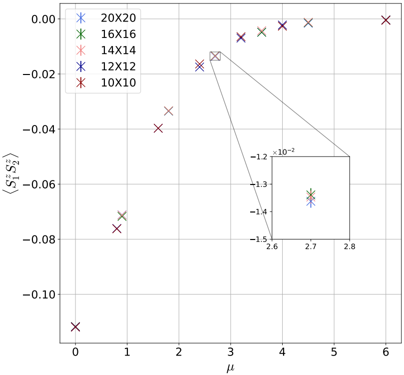


$\langle S_1^z S_j^z \rangle$ for different lattice sites at $\mu = 4.0$

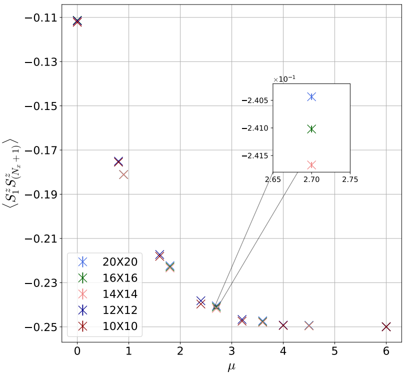
Results

Singlet Correlations



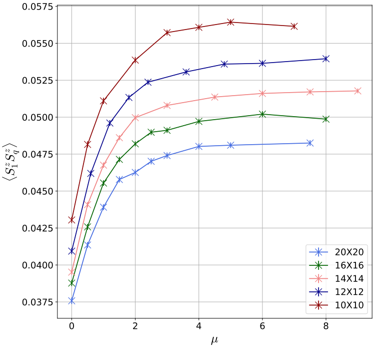


Spin-correlation variation with μ

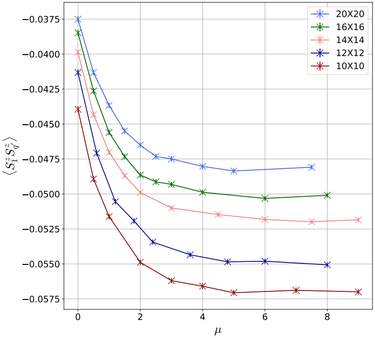


Spin-correlation variation with μ

Triplet Correlations



Spin correlation for distant spins $(q = \frac{N_x \times N_y}{2} \pm 1)$ belonging to same sub-lattice.



Spin correlation for distant spins $(q' = \frac{N_x \times N_y}{2} \pm 1)$ belonging to different sub-lattice.

Conclusion

Conclusion & Outlook

- Developed a framework to implement weak measurements in SSE.
- Singlet measurements disentangles the sites from the system where as triplet measurement enhances the long range order.

Outlook:

- Simulating measurements using different basis, to check if sign problem persists.
- Study the post-measurement entanglement structure of spins.
- Construct a general $M_{\{s\}}$ and look at the post-measurement correlations.